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Topic

Probability Distributions

Example

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- Consider a the random experiment be that of throwing a die.
- The six faces of the die can be treated as the six sample points in S={s_1 s_2 s_3 s_4 s_5 s_6}.
- Let $X(s_i)=i$, which transforms sample space to number line.
- This can be helpful in enquiring the probabilities such as

$$P[\{s:X(s)<=a_1\}]$$

$$P[\{s:X(s)<=c\}]$$

$$P[\{s: b_1<=X(s)<=b_1\}]$$

If distribution function (cumulative distribution function) of X() is known.

$$F_X(x) = P[\{s: X(s) \leq x\}]$$

Example

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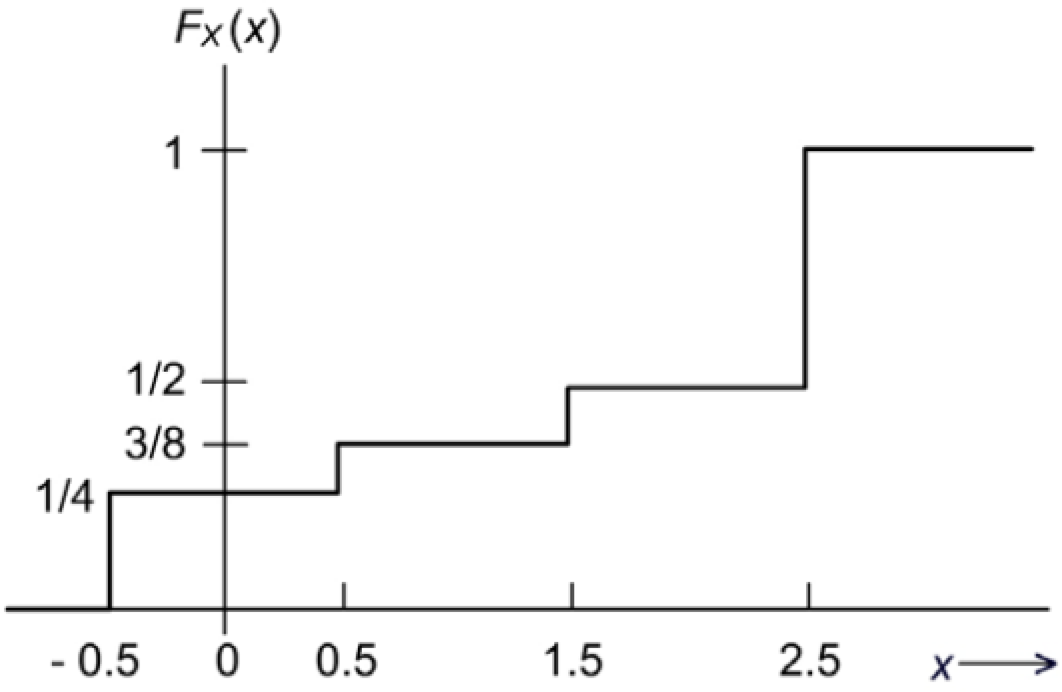
- Let $S=\{s_1\ s_2\ s_3\ s_4\}$ with $P(s_1)=1/4$, $P(s_2)=1/8$, $P(s_3)=1/8$, $P(s_4)=1/2$
- Let $X(s_i)=i-1.5$, where i=1,2,3,4, then CDF $F_X(x)$ will as follows.



Properties of CDF

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$F_X(\ )$ satisfies the following properties:

i) $F_X(x) \geq 0, \ -\infty < x < \infty$

ii) $F_X(-\infty) = 0$

iii) $F_X(\infty) = 1$

iv) If $a > b$, then $[F_X(a) - F_X(b)] = P[\{s: b < X(s) \leq a\}]$

v) If $a > b$, then $F_X(a) \geq F_X(b)$

Exercise-1

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Let S be a sample space with six sample points, s_1 to s_6. The events identified on S are the same as above, namely, $A = \{s_1, s_2\}$, $B = \{s_3, s_4, s_5\}$ and $C = \{s_6\}$ with $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$ and $P(C) = \frac{1}{6}$.

Let $Y(\ )$ be the transformation,

$$Y(s_i) = \begin{cases} 1, & i = 1, 2 \\ 2, & i = 3, 4, 5 \\ 3, & i = 6 \end{cases}$$

Show that $Y(\ )$ is a random variable by finding $F_Y(y)$. Sketch $F_Y(y)$.

Probability Density Function

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$$f_X(x) = \frac{d F_X(x)}{d x}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

Exercise-2

A random variable X has

$$F_X(x) = \begin{cases} 0 & , \quad x < 0 \\ K x^2 & , \quad 0 \le x \le 10 \\ 100K & , \quad x > 10 \end{cases}$$

(i) Find the constant K
(ii) Evaluate P[X ≤ 5] and P[5<X ≤7]
(iii) What is $f_x(X)$=?

Exercise-2: Solution

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i) $F_X(\infty) = 100K = 1 \Rightarrow K = \frac{1}{100}.$

ii) $P(x \leq 5) = F_X(5) = \left(\frac{1}{100}\right) \times 25 = 0.25$

$P(5 < X \leq 7) = F_X(7) - F_X(5) = 0.24$

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 &, \quad x < 0 \\ 0.02x &, \quad 0 \leq x \leq 10 \\ 0 &, \quad x > 10 \end{cases}$$

Exercise-3

Let $Y = \cos \pi X$, where

$$f_X(x) = \begin{cases} 1, & -\dfrac{1}{2} < x < \dfrac{1}{2} \\ 0, & \textit{otherwise} \end{cases}$$

Let us find $E[Y]$ and σ_Y^2.

Exercise-3:

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Let $Y = \cos \pi X$, where

$$f_X(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & otherwise \end{cases}$$

Let us find $E[Y]$ and σ_Y^2.

Solution

$$E[Y] = \int_{-1/2}^{1/2} \cos(\pi x)\, dx = \frac{2}{\pi} = 0.0636$$

$$E[Y^2] = \int_{-1/2}^{1/2} \cos^2(\pi x)\, dx = \frac{1}{2} = 0.5$$

Hence $\sigma_Y^2 = \frac{1}{2} - \frac{4}{\pi^2} = 0.96$ ♦

Covariance and Correlation

If X and Y are two random variables and $g(X,Y) = (X - m_X)(Y - m_Y)$

Where, $m_x = E[X]$ and $m_Y = E[Y]$

$$\lambda_{XY} = COV(X,Y) = E[(X - m_X)(Y - m_Y)]$$

Simplify above equations and find the formula for correlation.

Exercise-4

Let Y be the linear combination of the two random variables X_1 and X_2 as given below.

$$\mathrm{Y}=\boldsymbol{k_1 X_1}+\boldsymbol{k_2 X_2}$$

where, $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ are constants.

Let E[X_1]= m_1, E[X_2]= m_2, $\sigma_1^2$and σ_2^2 are variances of X_1 and X_2 respectively. The correlation coefficient between X_1 and X_2 is ρ_{12}.

Compute variance of Y, σ_Y^2=___________.

Exercise-5

Let X be a random variable which is uniformly distributed over the interval (0,1). Let Y be chosen from interval (0,X] according to the pdf

$$f(y/x) = \begin{cases} 1/x, & 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$$

Compute E$[Y^k]$ for a positive integer k.

Answer: $\dfrac{1}{(k+1)^2}$

Exercise-5

Consider a continuous random variable X, taking values in the range [0,255] and uniformly distributed. A sample of this random variable has been taken in the form of some images (5X5) in an environment with fixed properties such as illumination, environmental noise. etc. The pixel values in each image represents a random value of X.

What will be the expected mean of each observation in the sample?
What will be the expected variance of each observation in the sample?

Exercise-5

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Let (X1,Y1), (X2,Y2)……(Xn,Yn) are the n-numbers of paired observations for two random variables (X,Y). Assume $Y=aX^2+b$, where a and b are constants. Find the formula to compute the values of constants a and b by using the fitting based on MSE.

Book for practice