# MCA206: Statistics and Numerical Methods

## Topic

## Probability Distributions

# Example

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- Consider a the random experiment be that of throwing a die.
- The six faces of the die can be treated as the six sample points in $S=\{s_1\ s_2\ s_3\ s_4\ s_5\ s_6\}$.
- Let $X(s_i)=i$, which transforms sample space to number line.
- This can be helpful in enquiring the probabilities such as

$$P[\{s:X(s)<=a_1\}]$$

$$P[\{s:X(s)<=c\}]$$

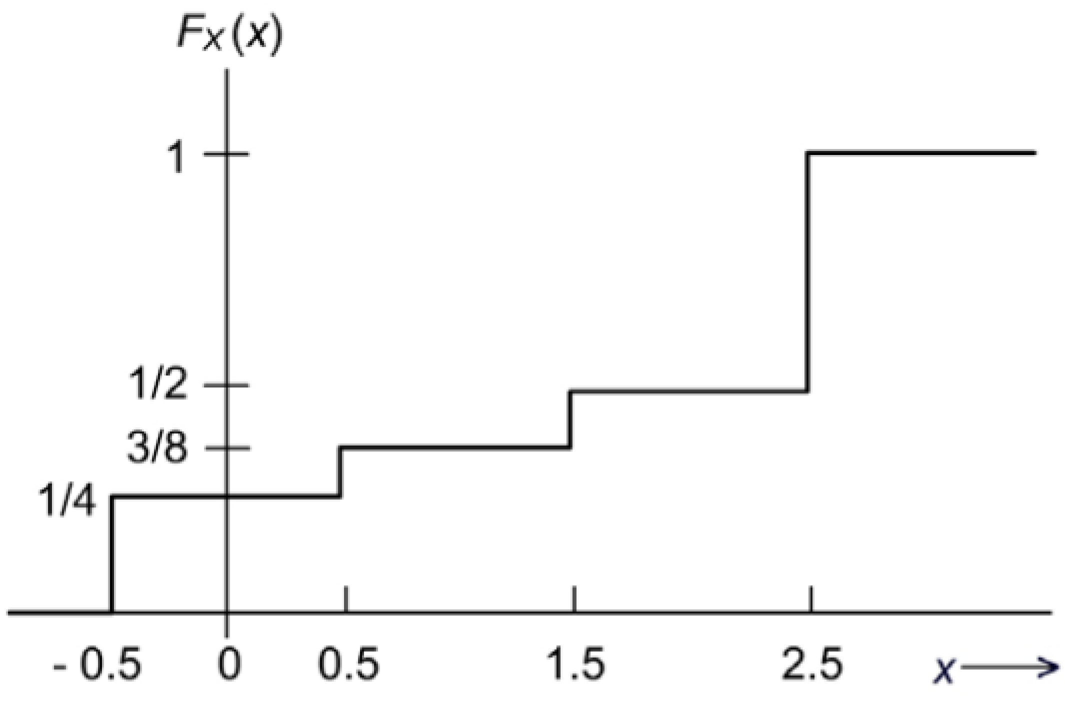
$$P[\{s:\ b_1<=X(s)<=b_1\}]$$

If distribution function (cumulative distribution function) of X() is known.

$$F_X(x)=P\left[\{s:X(s)\leq x\}\right]$$

# Example

- Let $S=\{s_1 \ s_2 \ s_3 \ s_4\}$ with $P(s_1)=1/4$, $P(s_2)=1/8$, $P(s_3)=1/8$, $P(s_4)=1/2$
- Let $X(s_i)=i-1.5$, where $i=1,2,3,4$, then CDF $F_X(x)$ will as follows.



# Properties of CDF

$F_X(\ )$ satisfies the following properties:

i) $F_X(x) \geq 0, \ -\infty < x < \infty$

ii) $F_X(-\infty) = 0$

iii) $F_X(\infty) = 1$

iv) If $a > b$, then $\left[F_X(a) - F_X(b)\right] = P\left[\{s : b < X(s) \leq a\}\right]$

v) If $a > b$, then $F_X(a) \geq F_X(b)$

# Exercise-1

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Let $\mathcal{S}$ be a sample space with six sample points, $s_1$ to $s_6$. The events identified on $\mathcal{S}$ are the same as above, namely, $A = \{s_1, s_2\}$, $B = \{s_3, s_4, s_5\}$ and $C = \{s_6\}$ with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$ and $P(C) = \frac{1}{6}$.

Let $Y(\ )$ be the transformation,

$$Y(s_i) = \begin{cases} 1, & i = 1, 2 \\ 2, & i = 3, 4, 5 \\ 3, & i = 6 \end{cases}$$

Show that $Y(\ )$ is a random variable by finding $F_Y(y)$. Sketch $F_Y(y)$.

# Probability Density Function

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

# Exercise-2

A random variable $X$ has

$$F_X(x) = \begin{cases} 0 & , \quad x < 0 \\ K x^2 & , \quad 0 \le x \le 10 \\ 100K & , \quad x > 10 \end{cases}$$

(i) Find the constant K

(ii) Evaluate $P[X \le 5]$ and $P[5 < X \le 7]$

(iii) What is $f_x(X)=?$

# Exercise-2: Solution

i) $F_X(\infty) = 100K = 1 \Rightarrow K = \dfrac{1}{100}.$

ii) $P(x \le 5) = F_X(5) = \left(\dfrac{1}{100}\right) \times 25 = 0.25$

$P(5 < X \le 7) = F_X(7) - F_X(5) = 0.24$

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 \quad , & x < 0 \\ 0.02x \, , & 0 \le x \le 10 \\ 0 \quad , & x > 10 \end{cases}$$

# Exercise-3

Let $Y = \cos \pi X$ , where

$$f_X(x) = \begin{cases} 1, & -\dfrac{1}{2} < x < \dfrac{1}{2} \\ 0, & otherwise \end{cases}$$

Let us find $E[Y]$ and $\sigma_Y^2$.

# Exercise-3:

Let $Y = \cos \pi X$, where

$$f_X(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \textit{otherwise} \end{cases}$$

Let us find $E[Y]$ and $\sigma_Y^2$.

# Solution

$$E[Y] = \int_{-1/2}^{1/2} \cos(\pi x)\, d x = \frac{2}{\pi} = 0.0636$$

$$E\left[Y^2\right] = \int_{-1/2}^{1/2} \cos^2(\pi x)\, d x = \frac{1}{2} = 0.5$$

Hence $\sigma_Y^2 = \frac{1}{2} - \frac{4}{\pi^2} = 0.96$ ♦

# Covariance and Correlation

If X and Y are two random variables and $g(X, Y) = (X - m_X)(Y - m_Y)$

Where, $m_x = E[X]$ and $m_Y = E[Y]$

$$\lambda_{XY} = COV(X, Y) = E[(X - m_X)(Y - m_Y)]$$

**Simplify above equations and find the formula for correlation.**

# Exercise-4

Let Y be the linear combination of the two random variables $X_1$ and $X_2$ as given below.

$$\mathrm{Y}=\boldsymbol{k_1 X_1}+\boldsymbol{k_2 X_2}$$

where, $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ are constants.

Let E[$X_1$]= $m_1$, E[$X_2$]= $m_2$, $\sigma_1^2$and $\sigma_2^2$ are variances of $X_1$ and $X_2$ respectively. The correlation coefficient between $X_1$ and $X_2$ is $\rho_{12}$.

Compute variance of Y, $\sigma_Y^2$=___________.

# Exercise-5

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Let X be a random variable which is uniformly distributed over the interval (0,1). Let Y be chosen from interval (0,X] according to the pdf

$$f(y/x) = \begin{cases} 1/x, & 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$$

Compute E$[Y^k]$ for a positive integer k.

**Answer:** $\frac{1}{(k+1)^2}$

# Exercise-5

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Consider a continuous random variable X, taking values in the range [0,255] and uniformly distributed. A sample of this random variable has been taken in the form of some images (5X5) in an environment with fixed properties such as illumination, environmental noise. etc. The pixel values in each image represents a random value of X.

What will be the expected mean of each observation in the sample?
What will be the expected variance of each observation in the sample?

# Exercise-5

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Let (X1,Y1), (X2,Y2)……(Xn,Yn) are the n-numbers of paired observations for two random variables (X,Y). Assume $Y=aX^2+b$, where a and b are constants. Find the formula to compute the values of constants a and b by using the fitting based on MSE.

# Book for practice